

Tilburg University

Non-additive beliefs and game theory

Eichberger, J.; Kelsey, D.

Publication date:
1994

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Eichberger, J., & Kelsey, D. (1994). *Non-additive beliefs and game theory*. (CentER Discussion Paper; Vol. 1994-10). CentER.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

CCBM
RR

Center

for

Economic Research

Discussion paper

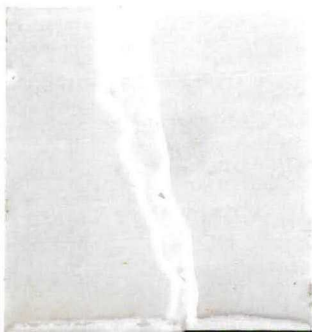


R 33

Handwritten text, possibly a title or date, is visible but illegible.



ENCLOSURE
OFFICIAL



Center
for
Economic Research

No. 9410

**NON-ADDITIVE BELIEFS
AND GAME THEORY**

by Jürgen Eichberger
and David Kelsey

January 1994



ISSN 0924-7815

NON-ADDITIVE BELIEFS AND GAME THEORY

Jürgen Eichberger* and David Kelsey**

December 1993

* Department of Economics, The University of Melbourne, Parkville,
Victoria 3052, Australia

** Department of Economics, The University of Birmingham, Birmingham,
B15 2TT, United Kingdom

NON-ADDITIVE BELIEFS AND GAME THEORY

Jürgen Eichberger and David Kelsey

ABSTRACT

This paper studies games where players' beliefs about their opponents' behaviour are modelled as non-additive probabilities. The concept of an *equilibrium under uncertainty* is introduced and discussed in the context of two-player games. Existence of such an equilibrium is demonstrated under usual conditions. For simple capacities equilibria under uncertainty approximate Nash equilibria as the degree of uncertainty vanishes. At the other extreme, as the degree of uncertainty goes to one, maximin equilibria appear. Finally, it can be shown that the use of non-additive probabilities can eliminate some implausible Nash equilibria.

Keywords: Uncertainty aversion, Choquet integral,
non-additive probabilities, equilibrium concepts

JEL Classification: C72, D81

Address for Correspondence: J. Eichberger, Department of Economics,
The University of Melbourne, Parkville,
Victoria 3052, Australia

NON-ADDITIVE BELIEFS AND GAME THEORY

Jürgen Eichberger and David Kelsey¹

1 Introduction

Experimental evidence indicates that preferences over uncertain acts cannot be represented by expected utility functionals. In particular, Ellsberg (1961) and others have pointed out that decision makers distinguish between situations under certainty and uncertainty in a way that cannot be represented by an additive probability distribution. Recently, Schmeidler (1989) and Gilboa (1987) have axiomatised a preference functional which does not imply that beliefs are represented by additive probabilities.

With the exception of Dow and Werlang (1991), no attempt has been made to investigate the implications of a decision theory without additive probabilities for game theory. Von Neumann and Morgenstern (1944) were among the first to axiomatise a decision theory for known lotteries and, since then, this has been the dominant paradigm for the analysis of games. In a Nash equilibrium, beliefs about the behaviour of opponents were supposed to coincide with the actual behaviour of these players. Interestingly, von Neumann and Morgenstern (1944) did not propose this equilibrium concept. They worked with a decision theory of complete ignorance assuming that players choose maximin strategies. According to this behavioural assumption agents consider the worst outcome for all strategies available and choose the strategy which yields the best among these worst outcomes. Von Neumann and Morgenstern (1944) could prove however that the value concept built on maximin behaviour coincides with the equilibrium concept for the class of zero-sum games.

¹ The first discussions on this topics between the authors took place while Jürgen Eichberger was visiting the University of Birmingham. The paper was written while Jürgen Eichberger was visiting the Center at Tilburg University. Jürgen Eichberger would like to thank both institutions for their hospitality which made this research possible. David Kelsey would like to acknowledge a travel grant from the School of Social Sciences, University of Birmingham. We would like to thank seminar participants at Queens University, The Institute of Advanced Studies in Vienna, the universities of Exeter, Osnabrück, and Birmingham for comments and discussion.

We will show in this paper how the theory of non-additive probabilities allows to reconcile the maximin approach with the Nash equilibrium concept for general games. The notion of a *degree of confidence* can be used to relate the two approaches. If the degree of confidence in a subjective probability representation of a belief is high players will behave like expected utility maximisers whereas they will act as maximin players for a high degree of uncertainty. It can be shown that, in general, equilibrium behaviour with non-additive beliefs will be as predicted by the Nash equilibrium concept if this degree of confidence is high. For low levels of confidence, however, equilibrium behaviour under uncertainty may be maximin behaviour.

A further result is the observation that there is a generic class of games where some lack of confidence in the probability assessment will rule out play of dominated strategies in equilibrium. This provides the possibility of a refinement of Nash equilibrium that is weaker than perfectness but still eliminates dominated strategy play in many games.

The paper is organised as follows. The next section introduces notation and provides a definition of an equilibrium concept with non-additive probabilities. Section 3 presents the concept of the degree of confidence, proves existence of an equilibrium and shows the dependence of equilibrium behaviour on this degree of confidence. Section 4 relates the equilibrium notion to the concept of maximin strategies and the final section shows how robustness against lack of confidence may be used as a refinement of Nash equilibrium.

2 Equilibrium under uncertainty

Consider a game $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$ with a finite player set I , finite pure strategy sets S_i for each player and payoff functions $p_i(s_i, s_{-i})$ which describe the payoff of player i if a strategy combination (s_i, s_{-i}) is played. The notation $s_{-i} := (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_I)$ indicates a strategy combination for all players except player i . It is convenient to denote by $S_{-i} := S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_I$ the set of strategy combinations which players other than i could choose.

Beliefs of players about opponents' behaviour are represented by non-additive probabilities (or capacities). A capacity assigns non-additive probability weights to subsets of a set of unknown states of the world. Let Ω be a set of states of the world, then one can formally define a capacity as follows.

Definition 2.1: (capacity)

A capacity on Ω is a real-valued function ν on the subsets of Ω which satisfies the following properties:

- a) $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$;
- b) $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$.

The capacity is called *convex* if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ holds.

Notice that, in contrast to an additive probability, it is not required that $\nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B)$. For a convex capacity it is possible that $\nu(A) + \nu(\Omega \setminus A) < 1$ holds, implying that not all probability mass is allocated to a set and its complement. Thus, one can define the support of a capacity either as the smallest set of states with measure one or the smallest set with a complement of measure zero². We will use the latter notion which can be formally defined as follows:

Definition 2.2: support of a capacity

The support of the capacity ν is an event $E \subseteq \Omega$ such that $\nu(\Omega \setminus E) = 0$ and $\nu(F) > 0$ for all F , $\Omega \setminus E \subset F$.

² Compare Dow and Werlang (1991) for a discussion and analysis of these two notions of a support of a measure.

Notice that the support of a capacity as defined in 2.2 is not unique in general. This is a problem which will be discussed in section 3. If the set of subsets of Ω has a measurable structure, then one can use the concept of a Choquet integral to determine the average value of measurable functions.

If Ω is finite and $f: \Omega \rightarrow \mathbb{R}$ is a function that associates values to states of the world, then $f(\Omega)$ is a finite set of real numbers. Denote by $f_1 = \max\{f(\omega) \mid \omega \in \Omega\}$ and by $Q_1 = \operatorname{argmax}\{f(\omega) \mid \omega \in \Omega\}$. Let $T_1 = Q_1$ and define iteratively,

$Q_j = \operatorname{argmax}\{f(\omega) \mid \omega \in \Omega \setminus T_{j-1}\}$, $f_j = \max\{f(\omega) \mid \omega \in \Omega \setminus T_{j-1}\}$, and $T_j = T_{j-1} \cup Q_j$. With the convention $Q_0 = \emptyset$, the Choquet integral can be formally defined as follows:

Definition 2.3: (Choquet integral)

For a finite set of states of the world Ω , the Choquet integral of a real-valued function f with regard to the capacity ν is

$$\mathfrak{J}(f, \nu) = \sum_{i=1}^n f_i \cdot [\nu(T_i) - \nu(T_{i-1})].$$

Each player $i \in I$ is supposed to hold a belief about her opponents' behaviour, a strategy combination in S_{-i} , which is represented by a capacity ν_i . Given this belief, the expected payoff from a strategy s_i can be determined by the Choquet integral $P_i(s_i, \nu_i) := \mathfrak{J}(p_i(s_i, \cdot), \nu_i)$. Players are assumed to choose a best response given their beliefs about their opponents' behaviour. Denote by $R_i(\nu_i) = \operatorname{argmax}\{P_i(s_i, \nu_i) \mid s_i \in S_i\}$ the best response correspondence of player i given beliefs ν_i .

To determine an equilibrium, it is necessary to relate players' beliefs to the actual behaviour. In a Nash equilibrium of a game where players' beliefs are represented by additive probabilities, expectations are assumed to be rational in the sense that each player chooses a (mixed) strategy which is a best response to the actual (mixed) strategies played by the opponents. Thus, beliefs and actual behaviour are usually not distinguished³. In the context of beliefs represented by non-additive probabilities, it is no longer possible to identify beliefs with actual behaviour. An important property of a Nash equilibrium is however the fact that all pure strategies

³ Crawford (1990) considers beliefs as additive probability distributions over mixed strategy spaces.

that are used in equilibrium with positive probability, i.e., that are in the support of the equilibrium mixed strategy, form best responses of the respective player.

The following definition of an equilibrium under uncertainty generalises the notion used in Dow and Werlang (1991) to games with an arbitrary finite number of players.

Definition 2.4: *equilibrium under uncertainty*

A belief system $(\nu_1^*, \dots, \nu_I^*)$ is an equilibrium under uncertainty if for all $i \in I$

$$\text{supp } \nu_i^* \subseteq \prod_{j \neq i \in I} R_j(\nu_j^*) .$$

In an equilibrium under uncertainty players beliefs as represented by the capacity ν^* put positive probability only on strategies that are best responses of the opponents given their equilibrium beliefs. Equilibrium beliefs can however no longer be interpreted as equilibrium mixed strategies because a player may believe that her opponents' strategy choices are correlated. The only consistency required in an equilibrium under uncertainty is each player chooses a pure strategy that is a best response given her beliefs and beliefs give positive probability weight only to best responses. The following example will illustrate this equilibrium concept.

Example 2.1:

Consider a general 2x2 matrix game.

		Player 2	
		t_1	t_2
Player 1	s_1	a_{11}, b_{11}	a_{12}, b_{12}
	s_2	a_{21}, b_{21}	a_{22}, b_{22}

In this case, beliefs of each player can be represented by two numbers indicating the probability that the opponent plays either of her two strategies:

$$\begin{aligned} \nu_1 &= (q_{t1}, q_{t2}) , & q_{t1} + q_{t2} &\leq 1, \\ \nu_2 &= (q_{s1}, q_{s2}) , & q_{s1} + q_{s2} &\leq 1. \end{aligned}$$

Notice that these probabilities need not sum to one. The support of these capacities is simply

$$\text{supp } \nu_1 = \begin{cases} \{t_1\} & \text{for } q_{t2} = 0 \\ \{t_2\} & \text{for } q_{t1} = 0 \\ \{t_1, t_2\} & \text{otherwise} \end{cases}$$

and

$$\text{supp } \nu_2 = \begin{cases} \{s_1\} & \text{for } q_{s2} = 0 \\ \{s_2\} & \text{for } q_{s1} = 0 \\ \{s_1, s_2\} & \text{otherwise} \end{cases}$$

Given beliefs ν_1 and ν_2 , it is straightforward to compute the following Choquet expected payoff functions:

$$P_1(s_i, \nu_1) = \begin{cases} a_{i1} \cdot q_{t1} + a_{i2} \cdot (1 - q_{t1}) & \text{for } a_{i1} > a_{i2} \\ a_{i1} & \text{for } a_{i1} = a_{i2} \\ a_{i2} \cdot q_{t2} + a_{i1} \cdot (1 - q_{t2}) & \text{for } a_{i1} < a_{i2} \end{cases}$$

$$P_2(t_i, \nu_2) = \begin{cases} b_{i1} \cdot q_{s1} + b_{2i} \cdot (1 - q_{s1}) & \text{for } b_{i1} > b_{2i} \\ b_{i1} & \text{for } b_{i1} = b_{2i} \\ b_{2i} \cdot q_{s2} + b_{i1} \cdot (1 - q_{s2}) & \text{for } b_{i1} < b_{2i} \end{cases}$$

Consider the case⁴ where the game has two Nash equilibria in pure and one in mixed strategies. This implies the following payoff parameter restrictions:

$$a_{11} > a_{12}, a_{21} < a_{22} \text{ and } b_{11} > b_{21}, b_{12} < b_{22}.$$

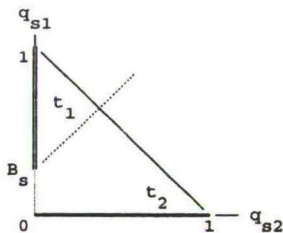
With these restrictions, one derives the following best response correspondences:

$$R_1(\nu_1) = \begin{cases} \{s_1\} & \text{for } q_{t1} > \frac{(a_{21} - a_{12})}{(a_{11} - a_{12})} + \frac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \cdot q_{t2} \\ \{s_1, s_2\} & \text{for } q_{t1} = \frac{(a_{21} - a_{12})}{(a_{11} - a_{12})} + \frac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \cdot q_{t2} \\ \{s_2\} & \text{for } q_{t1} < \frac{(a_{21} - a_{12})}{(a_{11} - a_{12})} + \frac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \cdot q_{t2} \end{cases}$$

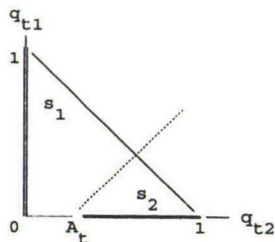
$$R_2(\nu_2) = \begin{cases} \{t_1\} & \text{for } q_{s1} > \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})} + \frac{(b_{22} - b_{12})}{(b_{11} - b_{21})} \cdot q_{s2} \\ \{t_1, t_2\} & \text{for } q_{s1} = \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})} + \frac{(b_{22} - b_{12})}{(b_{11} - b_{21})} \cdot q_{s2} \\ \{t_2\} & \text{for } q_{s1} < \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})} + \frac{(b_{22} - b_{12})}{(b_{11} - b_{21})} \cdot q_{s2} \end{cases}$$

⁴ The other generic payoff constellations can be analysed mutatis mutandis.

The following diagrams show the situation.



$$B_s = \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})}$$



$$A_t = \frac{(a_{12} - a_{21})}{(a_{22} - a_{21})}$$

Notice that the parameter restrictions given above guarantee that the denominators of B_s and A_t are positive. There is no restriction, however, on the sign of the numerators. The case⁵ depicted in the diagram corresponds to the additional parameter constraints $a_{12} - a_{21} > 0$ and $b_{12} - b_{21} > 0$.

A belief of player 1, ν_1 , is a pair (q_{t1}, q_{t2}) on or below the line $(0,1) - (1,0)$ in the right-hand diagram. Points in the left hand diagram represent different beliefs of player 2. The broken upward sloping lines indicate the loci where players are indifferent between their two strategies. These lines are upward sloping by the assumption on parameters made above but may have positive or negative intercepts. Beliefs above these lines make it a best response for each player to choose the first strategy; beliefs below these lines make the second strategies a best response. It is easy to check that the following beliefs form equilibria of the game:

- (i) (—): $\nu_1^* = (q_{t1}, 0)$, $0 \leq q_{t1} \leq 1$,
 $\nu_2^* = (q_{s1}, 0)$, $B_s \leq q_{s1} \leq 1$;
- (ii) (—): $\nu_1^* = (0, q_{t2})$, $A_t \leq q_{t2} \leq 1$,
 $\nu_2^* = (0, q_{s2})$, $0 \leq q_{s2} \leq 1$;
- (iii) (—): $\nu_1^* = (0, q_{t2})$, $0 \leq q_{t2} \leq A_t$,
 $\nu_2^* = (q_{s1}, 0)$, $0 \leq q_{s1} \leq B_s$;
- (iv) (—): $\nu_1^* = (q_{t1}, q_{t2})$, $A_t \leq q_{t2} \leq 1$,
 $\nu_2^* = (q_{s1}, q_{s2})$, $B_s \leq q_{s1} \leq 1$.

⁵ The other three cases are easily analysed by drawing the respective diagrams.

In an equilibrium of type (i), (s_1, t_1) will be played. This is a Nash equilibrium of this game under the assumed parameter restrictions. Similarly, type (ii) equilibria induce (s_2, t_2) as equilibrium play which corresponds to a second pure strategy Nash equilibrium of the game. Equilibria of type (iii) however lead to (s_1, t_2) being played which is not a Nash equilibrium. It is easy to check that (s_1, t_2) are the maximin strategies of the players given the payoff parameter restriction of the diagram. Finally, equilibria under uncertainty of type (iv) correspond to the mixed strategy Nash equilibrium because players are indifferent about their strategy choice. ■

Example 2.1 shows that, without further restrictions, there are many equilibria under uncertainty. Some of these equilibria induce the same equilibrium play as the Nash equilibria, while other equilibria are incompatible with Nash equilibrium play. The following section suggests a restriction on beliefs.

3 Simple capacities and the degree of confidence

In the game-theoretic context, beliefs have traditionally been represented by additive probability distributions. Identifying beliefs with probability distributions over states however makes it impossible to accommodate behaviour that distinguishes between cases where the decision maker knows the probability distribution over states and cases where s/he is ignorant about this distribution. Capacities allow us to make such a distinction.

General capacities have however little in common with probability distributions. In particular, the capacity of a set and the capacity of its complement need not sum to a number less or equal to one. Since capacities are supposed to represent beliefs, one restricts attention to convex capacities which have the property that $\nu(A) + \nu(\Omega \setminus A) \leq \nu(\Omega) = 1$ holds. Even convex capacities can have the undesirable property of a non-unique support. The following definition characterises a class of capacities which is easy to interpret as beliefs and which does not suffer from these problems.

Definition 3.1: A capacity ν is called *simple* if there exists an additive probability distribution π and a real number $\gamma \in (0,1)$ such that for all events $E \subseteq S$, $\nu(E) = \gamma \cdot \pi(E)$.

A *simple capacity*⁶ can be thought of as a contraction of an additive probability distribution. This could be interpreted as a lack of confidence that the decision-maker has in regard to her probability assessment. The parameter γ represents the *degree of confidence* that the agent has in the probabilistic assessment given by the additive probability distribution π . The smaller γ , the *degree of confidence*, the more uncertain is the agent about the situation. Uncertainty measured by γ can be distinguished from likelihood of a state represented by $\pi(\omega)$.

A further advantage of the representation of beliefs by simple capacities is the particularly simple form of the Choquet integral.

Proposition 3.1: Let ν be a simple capacity. For a finite set of states Ω ,

$$\mathfrak{J}(f, \nu) = \gamma \cdot \left[\sum_{i=1}^n f_i \cdot \pi(Q_i) \right] + (1 - \gamma) \cdot [\min\{f(\omega) \mid \omega \in \Omega\}].$$

Proof: From definition 2.3 and definition 3.1, one computes:

$$\begin{aligned} \mathfrak{J}(f, \nu) &= \sum_{i=1}^n f_i \cdot [\nu(T_i) - \nu(T_{i-1})] \\ &= \sum_{i=1}^{n-1} f_i \cdot \nu(Q_i) + f_n \cdot [\nu(\Omega) - \nu(T_{n-1})] = \sum_{i=1}^{n-1} f_i \cdot \gamma \cdot \pi(Q_i) + f_n \cdot [1 - \gamma \cdot \pi(\Omega \setminus Q_n)] \\ &= \gamma \cdot \sum_{i=1}^{n-1} f_i \cdot \pi(Q_i) + f_n \cdot [1 - \gamma \cdot (1 - \pi(Q_n))] = \gamma \cdot \sum_{i=1}^n f_i \cdot \pi(Q_i) + f_n \cdot [1 - \gamma]. \end{aligned}$$

Noting that $f_n = \min\{f(\omega) \mid \omega \in \Omega\}$ completes the proof. ■

Note that, for $Q_i = \{\omega_i\}$, $i = 1, \dots, n$, the Choquet integral simplifies to

$$\mathfrak{J}(f, \nu) = \gamma \cdot \left[\sum_{i=1}^n f_i \cdot \pi(s_i) \right] + (1 - \gamma) \cdot f_n.$$

⁶ Note that all simple capacities satisfy constant uncertainty aversion as defined in Dow and Werlang (1991). However, it is not the case that all capacities which display constant uncertainty aversion are simple. A special case of a simple capacity has been used by Dow and Werlang (1992).

The Choquet integral is a convex combination between the expected value of f given the additive measure π and the worst outcome f_n with weight $\gamma \in (0,1]$. The full weight of the non-additivity falls on the worst outcome in this case. If γ is close to zero, uncertainty dominates and the value of f is close to the worst outcome. Thus, a decision-maker who chooses an action to maximise the Choquet integral with respect to a simple capacity will

- maximise expected utility with probability distribution π for $\gamma = 1$, and
- choose a maximin action for $\gamma = 0$.

With beliefs characterised by a simple capacity with degree of confidence $\gamma \in (0,1)$, a player shows a decision-making behaviour which lies between expected utility maximisation and the extremely uncertainty-averse maximin behaviour.

Through a number of examples, Dow and Werlang (1991) have shown that the support of a capacity is, in general, not unique. An important property of simple capacities is therefore the fact that they have a unique support.

Proposition 3.2: For a simple capacity the support is unique and consists of all states with positive probability.

Proof: Let ν be a simple capacity on Ω . Since ν is simple there exists an additive probability π on Ω and $\gamma \in (0,1]$ such that for all $E \subseteq \Omega$, $\nu(E) = \gamma \cdot \pi(E)$. Let $B \subseteq \Omega$ denote the set of all states with positive measure π . By definition $\pi(\Omega \setminus B) = 0$, hence $\nu(\Omega \setminus B) = 0$. Let C be an event such that $\Omega \setminus B \subset C$. Then there exists $\omega' \in C \setminus (\Omega \setminus B) \subset B$. By monotonicity $\nu(C) \geq \nu(\{\omega'\}) = \gamma \cdot \pi(\{\omega'\}) > 0$. This demonstrates that B is the support of ν . ■

Applying the concept of a simple capacity to games, it can be shown that an equilibrium under uncertainty exists for any vector of exogenously given degrees of confidence that the players may have⁷.

Proposition 3.3: For any vector of parameters $\gamma := (\gamma_1, \dots, \gamma_I)$ there exists a Nash equilibrium under uncertainty.

⁷ Proposition 3.3 confirms a conjecture of Dow and Werlang (1991) that a Nash equilibrium under uncertainty will exist for any exogenously given bound on the uncertainty aversion of players.

Proof:

Let $m_i(s_i) := \min\{p_i(s_i, s_{-i}) \mid s_{-i} \in S_{-i}\}$ be the minimum payoff a player obtains from playing strategy s_i . Define new payoff functions

$$\psi_i(s_i, s_{-i}) := p_i(s_i, s_{-i}) + [(1-\gamma_i)/\gamma_i] \cdot m_i(s_i).$$

The new game $\Gamma' = (I, (S_i)_{i \in I}, (\psi_i)_{i \in I})$ is well-defined and has a Nash equilibrium $\pi^* = (\pi_1^*, \dots, \pi_I^*)$ in mixed strategies. Thus, for all $i \in I$ and all $s_i^* \in S_i$ with $\pi_i^*(s_i^*) > 0$,

$$\sum_{s_{-i} \in S_{-i}} \psi_i(s_i^*, s_{-i}) \cdot \pi^*(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \psi_i(s_i, s_{-i}) \cdot \pi^*(s_{-i})$$

for all $s_i \in S_i$ must hold, where

$$\pi(s_{-i}) := \pi_1(s_1) \cdot \pi_2(s_2) \cdot \dots \cdot \pi_{i-1}(s_{i-1}) \cdot \pi_{i+1}(s_{i+1}) \cdot \dots \cdot \pi_I(s_I)$$

denotes the probability that s_{-i} is played.

Let $\nu_i^*(s_{-i}) := \gamma_i \cdot \pi^*(s_{-i})$ for all $s_{-i} \in S_{-i}$ denote the belief of player $i \in I$ with constant uncertainty aversion γ_i . It will be shown that $\nu^* = (\nu_1^*, \dots, \nu_I^*)$ forms an equilibrium under uncertainty for the original game with constant uncertainty aversion parameters γ .

Consider $s_{-i}^* \in \text{supp } \nu_i^*$. By proposition 3.2, $\nu_i^*(s_{-i}^*) = \gamma_i \cdot \pi^*(s_{-i}^*) = \gamma_i \cdot [\pi_1^*(s_1^*) \cdot \pi_2^*(s_2^*) \cdot \dots \cdot \pi_{i-1}^*(s_{i-1}^*) \cdot \pi_{i+1}^*(s_{i+1}^*) \cdot \dots \cdot \pi_I^*(s_I^*)] > 0$.

Hence, for $j \neq i$, $\pi_j^*(s_j^*) > 0$ and

$$s_j^* \in \arg\max_{s_j \in S_j} \left\{ \sum_{s_{-j} \in S_{-j}} \psi_j(s_j, s_{-j}) \cdot \pi^*(s_{-j}) \mid s_j \in S_j \right\} \quad (*)$$

by the definition of a Nash equilibrium.

Since the capacity ν_j^* is simple,

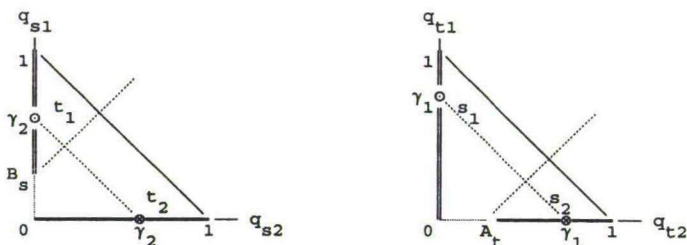
$$\begin{aligned} P_j(s_j^*, \nu_j^*) &:= \sum_{s_{-j} \in S_{-j}} P_j(s_j^*, s_{-j}) \cdot \nu_j^*(s_{-j}) + (1-\gamma_j) \cdot m_j(s_j^*) \\ &= \gamma_j \cdot \sum_{s_{-j} \in S_{-j}} [p_j(s_j^*, s_{-j}) + [(1-\gamma_j)/\gamma_j] \cdot m_j(s_j^*)] \cdot \pi^*(s_{-j}) \\ &= \gamma_j \cdot \sum_{s_{-j} \in S_{-j}} \psi_j(s_j^*, s_{-j}) \cdot \pi^*(s_{-j}) \\ &\geq \gamma_j \cdot \sum_{s_{-j} \in S_{-j}} \psi_j(s_j, s_{-j}) \cdot \pi^*(s_{-j}) = P_j(s_j, \nu_j^*) \end{aligned}$$

follows from (*). This proves that ν^* is an equilibrium under uncertainty. ■

For the case of 2x2 matrix games which was discussed in example 2.1, the set of equilibria for a given degree of confidence of the players can be given a simple geometric interpretation.

Example 3.1: (example 2.1 resumed)

Recall the diagrammatic representation of beliefs in example 2.1. The exogenously given degrees of confidence γ_1 and γ_2 require beliefs to lie on the lines $(0, \gamma_1) - (\gamma_1, 0)$ and $(0, \gamma_2) - (\gamma_2, 0)$ in the respective diagrams.



If γ_1 and γ_2 are as given in the diagram above, then one checks easily that the following three equilibria under uncertainty exist:

- (i) $(\circ \circ)$: $\nu_1^* = (\gamma_1, 0)$, $\nu_2^* = (\gamma_2, 0)$;
- (ii) $(\bullet \bullet)$: $\nu_1^* = (0, \gamma_1)$, $\nu_2^* = (0, \gamma_2)$;
- (iii) $\nu_1^* = (q_{t1}, q_{t2})$, $\nu_2^* = (q_{s1}, q_{s2})$ with

$$q_{t1} + q_{t2} = \gamma_1, \quad q_{t1} = \frac{(a_{21} - a_{12})}{(a_{11} - a_{12})} + \frac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \cdot q_{t2},$$

$$q_{s1} + q_{s2} = \gamma_2, \quad q_{s1} = \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})} + \frac{(b_{22} - b_{12})}{(b_{11} - b_{21})} \cdot q_{s2}. \quad \blacksquare$$

For high degrees of confidence, equilibrium behaviour for the pure strategy equilibria ((i) and (ii)) will be exactly as in the respective Nash equilibria and the third equilibrium will have a support which coincides with the support of the mixed strategy Nash equilibrium. On the other hand, it is easy to see in this example that low⁸ values of γ_1 and γ_2 lead to maximin behaviour in an equilibrium under uncertainty. The following section

⁸ For $\gamma_1, \gamma_2 < \min \{A_t, B_s\}$, behaviour in an equilibrium under uncertainty is maximin behaviour.

investigates whether this relationship between Nash equilibrium and equilibrium under uncertainty holds for general games as well.

4 Equilibrium under uncertainty and Nash equilibrium

This section relates equilibria under uncertainty with simple capacities to Nash equilibria on the one hand and maximin strategies on the other. It is well-known that, for zero-sum games, every Nash equilibrium strategy combination is a maximin strategy combination. This identity of prudent and equilibrium behaviour is however lost for general games. The concept of an equilibrium under uncertainty with simple capacities makes it possible to relate these two traditional methods of determining the solution of a game (compare Moulin (1986)). Indeed, the less confidence players have in their beliefs about opponents' behaviour the more likely the pure strategy chosen in equilibrium will be a maximin strategy combination. If players are confident about their beliefs, then they will play as in a Nash equilibrium. The notion of an equilibrium under uncertainty is however general enough to allow for some players to play maximin strategies and for others to play best responses.

First, low degrees of confidence in the probability assessment are considered. Let

$$m_i(s_i) := \min\{p_i(s_i, s_{-i}) \mid s_{-i} \in S_{-i}\}$$

be the worst payoff player i can obtain from playing strategy s_i and let

$$M_i := \operatorname{argmax}\{m_i(s_i) \mid s_i \in S_i\}.$$

be the strategies of player i that maximise these worst payoffs. Note that any strategy combination $s = (s_1, \dots, s_I) \in \prod_{i \in I} M_i$ is a maximin strategy combination. For two-player zero-sum games, any such strategy combination leads to the same payoff vector.

Definition 4.1:

An equilibrium under uncertainty with simple capacities $\nu^* = (\nu_1^*, \dots, \nu_I^*)$ induces maximin play if, for all $i \in I$,

$$R_i(\nu_i^*) = M_i.$$

The following result shows that, for low degrees of confidence, equilibria under uncertainty are maximin strategy combinations.

Proposition 4.1: Given beliefs that are represented by simple capacities, there exists $\epsilon > 0$ such that, for $\gamma_i \in [0, \epsilon]$ for all $i \in I$, every equilibrium under uncertainty induces maximin play.

Proof:

Firstly, for $M_i = S_i$, $P_i(s_i, s_{-i}) = P_i(s'_i, s'_{-i})$ for all $s'_i \in S_i$ and all $s'_{-i} \in S_{-i}$. Hence, $P_i(s_i, \nu_i^*) = P_i(s'_i, \nu_i^*)$ for all $s'_i \in S_i$ and $R_i(\nu_i^*) = S_i$.

Secondly, for $M_i \neq S_i$, consider any strategy $s_i \in R_i(\nu_i^*)$ such that $s_i \notin M_i$, then $m_i(s'_i) > m_i(s_i)$ for all $s'_i \in M_i$ and

$$\begin{aligned} P_i(s_i, \nu_i^*) &= \gamma_i \cdot \left[\sum_{s_{-i} \in S_{-i}} P_i(s_i, s_{-i}) \cdot \alpha_i(s_{-i}) \right] + (1 - \gamma_i) \cdot m_i(s_i) \\ &\geq \gamma_i \cdot \left[\sum_{s_{-i} \in S_{-i}} P_i(s'_i, s_{-i}) \cdot \alpha_i(s_{-i}) \right] + (1 - \gamma_i) \cdot m_i(s'_i) = P_i(s'_i, \nu_i^*), \end{aligned}$$

where α_i is an additive probability distribution on S_{-i} such that $\nu_i^*(s_{-i}) = \gamma_i \cdot \alpha_i(s_{-i})$ for all $s_{-i} \in S_{-i}$.

Since $m_i(s'_i) > m_i(s_i)$ holds, there is a positive ϵ_i small enough such that $P_i(s'_i, \nu_i^*) > P_i(s_i, \nu_i^*)$ for all $\gamma_i \leq \epsilon_i$ holds. For such a γ_i , $s_i \in R_i(\nu_i^*) \cap M_i$. Thus, $\epsilon = \min\{\epsilon_i \mid i \in I\}$ provides an upper bound on the degree of confidence such that $R_i(\nu_i^*) = M_i$. ■

Example 3.1 suggests that, for high degrees of confidence, equilibrium play will resemble a Nash equilibrium. There is, however, a complication if more than two players are considered. For games with three or more agents, a player may believe that the opponents' behaviour is correlated. In a Nash equilibrium, however, players' behaviour must be uncorrelated. This motivates the following definition.

Definition 4.2:

Player i believes that the opponents act *independently* if

$$\nu_i(s_{-i}) = \prod_{j \neq i} \nu_i^j(s_j)$$

holds for some simple capacities $\nu_i^j(s_j) = \gamma_i^j \cdot \pi_i^j(s_j)$, $j \neq i$.

It is one of the advantages of simple capacities that one can define product measures using the additive structure of the defining probability distribution $\pi_i = (\pi_i^1, \dots, \pi_i^{i-1}, \pi_i^{i+1}, \dots, \pi_i^I)$. For general capacities, this is usually impossible.

Definition 4.3:

A system of beliefs that all players act independently is called *consistent* if $\nu_i^j = \nu_j^j$ for all $i, j \in I$, $i \neq j$.

An equilibrium under uncertainty leads to actual play which is equivalent to Nash equilibrium play if the equilibrium strategy combinations could result from a Nash equilibrium.

Definition 4.4:

An equilibrium under uncertainty with simple capacities $\nu^* = (\nu_1^*, \dots, \nu_I^*)$ induces Nash equilibrium play if there is a Nash equilibrium (in mixed strategies) $\mu^* = (\mu_1^*, \dots, \mu_I^*)$ such that, for all $i \in I$,

$$\text{supp } \mu_i^* \subseteq R_i(\nu_i^*) \quad \text{with} \quad \nu_i^*(s_{-i}) = \prod_{j \neq i} \mu_j^*(s_j) \quad \text{for all } s_{-i} \in S_{-i}.$$

The following proposition relates play in an equilibrium under uncertainty for high degrees of confidence to Nash equilibria.

Proposition 4.2: If all players believe that their opponents act independently and if their beliefs are consistent, then there exists $\epsilon > 0$ such that, for $\gamma_i^j \in [1 - \epsilon, 1]$, $i, j \in I$, $i \neq j$, every equilibrium under uncertainty induces Nash equilibrium play.

Proof:

Consider an equilibrium under uncertainty ν^* where all players believe that their opponents act independently and where beliefs are consistent, then there exists $\pi = (\pi^1, \dots, \pi^I)$ such, that for all $i \in I$ and all $s_{-i} \in S_{-i}$,

$$\nu_i^*(s_{-i}) = \prod_{j \neq i} \gamma_i^j \cdot \pi_j^j(s_j), \quad i \neq j.$$

By proposition 3.2, $\text{supp } \nu_i^*(s_{-i}) = \prod_{j \neq i} \text{supp } \pi_j^j(s_j)$, for all $i \in I$. Since ν^* is an equilibrium under uncertainty, one has

$$s_j \in \text{supp } \pi_j^j \text{ implies } s_j \in R_j(\nu_j^*) \text{ for all } j \in I.$$

Therefore, with $\gamma_j := \prod_{i \neq j} \gamma_j^i$ and $\pi(s_{-j}) := \prod_{i \neq j} \pi^i(s_i)$,

$$\begin{aligned} P_j(s_j, \nu_j^*) &= \gamma_j \cdot \left[\sum_{s_{-j} \in S_{-j}} p_j(s_j, s_{-j}) \cdot \pi(s_{-j}) \right] + (1 - \gamma_j) \cdot m_j(s_j) \\ &\geq \gamma_j \cdot \left[\sum_{s_{-j} \in S_{-j}} p_j(s'_j, s_{-j}) \cdot \pi(s_{-j}) \right] + (1 - \gamma_j) \cdot m_j(s'_j) = P_j(s'_j, \nu_j^*) \end{aligned}$$

for all $s'_j \in S_j$.

Such an equilibrium under uncertainty exists for all degrees of confidence γ (proposition 3.3). Hence, there must be an ϵ_j such that, for all $\gamma_j \geq 1 - \epsilon_j$ and the associated probability distribution π ,

$$\left[\sum_{s_{-j} \in S_{-j}} p_j(s_j, s_{-j}) \cdot \pi(s_{-j}) \right] \geq \left[\sum_{s_{-j} \in S_{-j}} p_j(s'_j, s_{-j}) \cdot \pi(s_{-j}) \right].$$

Thus, π is a Nash equilibrium (in mixed strategies) and $\mu^* := \pi$ satisfies the claim of the proposition. ■

In two-player games, the consistency and independence condition of proposition 4.2 are trivially satisfied.

5 Robustness against lack of confidence

As the previous section has shown, for γ_i close to 1 for all $i \in I$, equilibrium behaviour under uncertainty is the same as in a Nash equilibrium. This raises the question whether all Nash equilibria have a nearby equilibrium under uncertainty with the same equilibrium play. If this is the case for high degrees of confidence, then one can consider the Nash equilibrium as robust against some lack of confidence. On the other hand, if there are games where, for some Nash equilibrium, there is no equilibrium under uncertainty inducing this Nash equilibrium play, then such a Nash equilibrium is not robust against some lack of confidence.

The following proposition shows that there are games with Nash equilibria whose equilibrium play occurs in no equilibrium under uncertainty.

Proposition 5.1:

Consider games such that $m_i(s_i) \neq m_i(s'_i)$ for all $s_i \neq s'_i$ holds. If beliefs are represented by simple capacities, then an equilibrium under uncertainty does not use dominated strategies.

Proof:

Let $\nu^* = (\nu_1^*, \dots, \nu_I^*)$ be an equilibrium under uncertainty. Since ν_j^* are simple capacities there must be an additive probability distribution α^j on S_{-j} such that $\nu_j^* = \gamma_j \cdot \alpha^j$ for all $j \in I$. Let s_i be a strategy of some player $i \neq j$ which forms part of a strategy combination $s_{-j} \in \text{supp } \nu_j^*$. By the definition of an equilibrium under uncertainty, one has

$$\begin{aligned}
P_i(s_i, \nu_i^*) &= \gamma_i \cdot \left[\sum_{s_{-i} \in S_{-i}} P_i(s_i, s_{-i}) \cdot \alpha^i(s_{-i}) \right] + (1-\gamma_i) \cdot m_i(s_i) \\
&\geq \gamma_i \cdot \left[\sum_{s_{-i} \in S_{-i}} P_i(s'_i, s_{-i}) \cdot \alpha^i(s_{-i}) \right] + (1-\gamma_i) \cdot m_i(s_i) = P_i(s'_i, \nu_i^*)
\end{aligned}$$

for all $s'_i \in S_i$.

Suppose that s_i is dominated by strategy s_i'' , i.e. $p_i(s_i'', s_{-i}) \geq p_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ with strict inequality for some s_{-i} . Clearly,

$$\sum_{s_{-i} \in S_{-i}} P_i(s_i'', s_{-i}) \cdot \alpha^i(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} P_i(s_i, s_{-i}) \cdot \alpha^i(s_{-i})$$

and

$$\begin{aligned}
P_i(s_i'', \nu_i^*) &= \gamma_i \cdot \left[\sum_{s_{-i} \in S_{-i}} P_i(s_i'', s_{-i}) \cdot \alpha^i(s_{-i}) \right] + (1-\gamma_i) \cdot m_i(s_i'') \\
&> \gamma_i \cdot \left[\sum_{s_{-i} \in S_{-i}} P_i(s_i, s_{-i}) \cdot \alpha^i(s_{-i}) \right] + (1-\gamma_i) \cdot m_i(s_i) = P_i(s_i, \nu_i^*) \text{ in}
\end{aligned}$$

this case because

$$m_i(s_i'') = p_i(s_i'', \tilde{s}_{-i}) \geq p_i(s_i, \tilde{s}_{-i}) \geq m_i(s_i).$$

By the premise $m_i(s_i) \neq m_i(s_i'')$ and, hence,

$$m_i(s_i'') > m_i(s_i).$$

This contradicts the required optimality of s_i . ■

We conclude this section with an example which illustrates the necessity of the condition $m_i(s_i) \neq m_i(s_i'')$.

Example 5.1:

Consider the following 2x2 matrix game.

		Player 2	
		t_1	t_2
Player 1	s_1	1, 1	0, -1
	s_2	-1, 0	0, 0

It is easy to check that

$$m_1(s_1) = 0 \neq m_1(s_2) = -1 \quad \text{and}$$

$$m_2(t_1) = 0 \neq m_2(t_2) = -1$$

holds. There are two Nash equilibria in pure strategies (s_1, t_1) and (s_2, t_2) . The second Nash equilibrium (s_2, t_2) uses dominated strategies. According to proposition 5.1 there is therefore no equilibrium under uncertainty with this support. To check this claim, consider player 1. The Choquet expected payoff of strategy 1 is $P_1(s_1, \nu_1) = q_{t1}$ and for strategy 2 $P_1(s_2, \nu_1) = q_{t2} - 1$. For any degree of confidence less than 1, it is impossible that

$$P_1(s_1, \nu_1) = q_{t1} \leq P_1(s_2, \nu_1) = q_{t2} - 1 < 1.$$

Since this is necessary for an equilibrium under uncertainty with support (s_2, t_2) , no equilibrium under uncertainty can have this Nash equilibrium play.

Consider on the other hand the following modification of this game.

		Player 2	
		t_1	t_2
Player 1	s_1	1, 1	0, 0
	s_2	0, 0	0, 0

Notice that this modification did not change the set of pure Nash equilibria nor the fact that s_2 and t_2 are dominated strategies. The condition of proposition 5.1 is however no longer satisfied, since

$$m_1(s_1) = m_1(s_2) = 0 \quad \text{and} \quad m_2(t_1) = m_2(t_2) = 0.$$

The condition on the Choquet expected utilities of player 1 for an equilibrium under uncertainty is now

$$P_1(s_1, \nu_1) = q_{t1} \leq P_1(s_2, \nu_1) = 0,$$

which can be satisfied for $q_{t1} = 0$. Similarly, one can show that for $q_{s1} = 0$ player 2 may find it optimal to choose t_2 . Thus, in this case, there is an equilibrium under uncertainty compatible with this Nash equilibrium play. ■

Proposition 5.1 and the example show that 'robustness against some lack of confidence', does not coincide with refinements like perfectness or iterated deletion of dominated strategies. The necessary condition of proposition 5.1, that there be no ties between the minimum payoffs achieved with different pure strategies of a player, is however not a generic property for

two-player games with finite pure strategy sets. Thus, there may be a closer relationship between these refinements for generic games.

The paper concludes with an example which illustrates how the notion of a degree of confidence may be used to derive new results in economic applications.

Example 5.2:

Consider n communities that have to cooperate in the prevention of pollution. The quality of the environment x depends on the contribution of each community v_i , $i = 1, \dots, n$, according to the production function $x = \min\{v_i \mid i = 1, \dots, n\}$. Thus, the minimal care taken determines the overall outcome. All communities have the same preferences over environmental quality and effort given by the utility function $u_i(x, v_i) = 2 \cdot x - v_i$. To simplify the exposition assume that effort levels can only take the values 1 and 2.

For the case of two communities the following payoff matrix arises from this scenario.

		Player 2	
		2	1
Player 1	2	2, 2	0, 1
	1	1, 0	1, 1

It is straightforward to compute the following Choquet integral of player i :

$$\mathfrak{J}(p_i(2, \cdot), \gamma_i \cdot \pi_i) = \gamma_i \cdot [2 \cdot \pi_i(2) + 0 \cdot \pi_i(1)] + (1 - \gamma_i) \cdot 0 = 2 \cdot \gamma_i \cdot \pi_i(2)$$

$$\mathfrak{J}(p_i(1, \cdot), \gamma_i \cdot \pi_i) = 1.$$

Clearly, player i will choose effort level 2 if and only if $\gamma_i \cdot \pi_i(2) \geq 0.5$.

For an equilibrium under uncertainty where players believe that the opponent plays never strategy 1, i.e. with $\pi_i = (\pi_i(2), \pi_i(1)) = (1, 0)$, $\gamma_i \geq 0.5$ is a necessary condition. In fact, it is easy to see that for $\gamma_i < 0.5$ there is a unique equilibrium under uncertainty where both players choose a low effort level, $\pi_i = (0, 1)$.

Consider now the case where the number of players increases. For three players the Choquet integral of player 1 becomes

$$\mathfrak{J}(p_1(2, \cdot), \nu_1) = \gamma_1^2 \cdot \gamma_1^3 \cdot [2 \cdot \pi_1^2(2) \cdot \pi_1^3(2)] \quad \text{and} \quad \mathfrak{J}(p_1(1, \cdot), \nu_1) = 1.$$

Assuming all players believe that opponents play independently and beliefs that are consistent, an equilibrium under uncertainty in which (2,2) is played requires now that $\gamma_2 \cdot \gamma_3 \geq 0.5$ holds. Assuming further the same degree of confidence for each player, $\gamma \geq \sqrt{0.5} > 0.5$ is required for coordination on the equilibrium with contribution 2. Thus, one can see that an increase in the number of participants reduces the possibility of the good coordination equilibrium (2,2).

In fact, it is easy to check that, with n players, $\gamma^{n-1} < 0.5$ implies a unique equilibrium under uncertainty with contribution level 1. Thus, for any lack of confidence $1 - \gamma > 0$, no matter how small, there is a number of players N large enough to make contributing only one unit the only equilibrium under uncertainty, since $\lim_{n \rightarrow \infty} \sqrt[n-1]{0.5} = 1$. One can therefore conclude that the larger the number of players of this game the more likely is the equilibrium with the smallest possible contribution level. ■

6 Concluding remarks

This paper has introduced an equilibrium concept for games with a finite number of players whose beliefs are represented by non-additive probabilities. In general, relaxing the restrictions imposed on beliefs by the probabilistic nature of mixed strategies and the consistency of beliefs required in a Nash equilibrium will increase the number of equilibria substantially. The minimal deviation from a probabilistic representation of beliefs that is implied by the concept of a simple capacity makes it possible to parametrise the degree of deviation. This degree of deviation from a representation of beliefs by a probability distribution can be interpreted as a player's lack confidence in the assessment of the opponents' play.

With the help of the concept of the degree of confidence, it is possible to parametrise equilibria under uncertainty. It could be shown that, without further assumptions, equilibria under uncertainty will coincide with maximin strategies of the players if the degree of confidence of all players is low.

On the other hand, for high degrees of confidence, equilibria under uncertainty are similar to Nash equilibria only if all players believe that their opponents act independently and if these beliefs are mutually consistent. This result shows the importance of these two properties for the Nash equilibrium concept. In addition, the notion of an equilibrium under uncertainty allows the analyst of a game to test the robustness of Nash equilibrium in regard to these two implicit assumptions.

It could be shown as well that robustness against small degrees of uncertainty provides a refinement of the Nash equilibrium concept that does not coincide with perfectness or robustness against iterated deletion of dominated strategies. In this area, however, more research is necessary.

We conclude the paper with a brief discussion of other game-theoretic equilibrium concepts where the assumption of expected-utility maximising agents is abandoned. Crawford (1990) introduces the notion of an *equilibrium in beliefs*. In Crawford (1990), beliefs are probability distributions over mixed strategies, not capacities over pure strategies as in this paper. Nevertheless, there is a noteworthy relationship between the concept of an equilibrium in beliefs and an equilibrium under uncertainty. An equilibrium in beliefs requires a player's belief to be concentrated on the set of best-reply mixed strategies of the other player given that player's beliefs. In this regard, equilibria in beliefs are similar to equilibria under uncertainty. In particular, strategies that are actually played need not coincide with the strategy that the opponent expects as long as it is part of the support of the opponent's belief. Since Crawford (1990) introduces the concept of an equilibrium in beliefs for two-player games only, the issues of players' believing that their opponents act independently and of consistency of beliefs do not arise.

References

- Crawford, V.P. (1990). "Equilibrium without Independence". *Journal of Economic Theory* 50, 127-154.
- Dow, J., Werlang, S.R.d.C. (1991). "Nash Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction". *unpublished paper*, London Business School.
- Dow, J., Werlang, S.R.d.C. (1992). "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio". *Econometrica* 60, 197-204.
- Ellsberg, D. (1961). "Risk, Ambiguity and the Savage Axioms". *Quarterly Journal of Economics*, 75, 643-669.
- Gilboa, I. (1987). "Expected Utility Theory with Purely Subjective Non-Additive Probabilities". *Journal of Mathematical Economics*, 16, 65-88.
- Moulin, H. (1986). *Game Theory for the Social Sciences* (2nd and revised edition). New York: New York University Press.
- Schmeidler, D. (1989). "Subjective Probability and Expected Utility without Additivity". *Econometrica*, 57, 571-587.
- Von Neumann, J., Morgenstern, O. (1947). *Theory of Games and Economic Behavior*. 2nd edition. Princeton, N.J.: Princeton University Press.

Discussion Paper Series, CentER, Tilburg University, The Netherlands:

(For previous papers please consult previous discussion papers.)

No.	Author(s)	Title
9303	D. Karotkin and S. Nitzan	Some Peculiarities of Group Decision Making in Teams
9304	A. Lusardi	Euler Equations in Micro Data: Merging Data from Two Samples
9305	W. Güth	A Simple Justification of Quantity Competition and the Cournot-Oligopoly Solution
9306	B. Peleg and S. Tijs	The Consistency Principle For Games in Strategic Form
9307	G. Imbens and A. Lancaster	Case Control Studies with Contaminated Controls
9308	T. Ellingsen and K. Wärneryd	Foreign Direct Investment and the Political Economy of Protection
9309	H. Bester	Price Commitment in Search Markets
9310	T. Callan and A. van Soest	Female Labour Supply in Farm Households: Farm and Off-Farm Participation
9311	M. Pradhan and A. van Soest	Formal and Informal Sector Employment in Urban Areas of Bolivia
9312	Th. Nijman and E. Sentana	Marginalization and Contemporaneous Aggregation in Multivariate GARCH Processes
9313	K. Wärneryd	Communication, Complexity, and Evolutionary Stability
9314	O.P. Attanasio and M. Browning	Consumption over the Life Cycle and over the Business Cycle
9315	F. C. Drost and B. J. M. Werker	A Note on Robinson's Test of Independence
9316	H. Hamers, P. Borm and S. Tijs	On Games Corresponding to Sequencing Situations with Ready Times
9317	W. Güth	On Ultimatum Bargaining Experiments - A Personal Review
9318	M.J.G. van Eijss	On the Determination of the Control Parameters of the Optimal Can-order Policy
9319	S. Hurkens	Multi-sided Pre-play Communication by Burning Money
9320	J.J.G. Lemmen and S.C.W. Eijffinger	The Quantity Approach to Financial Integration: The Feldstein-Horioka Criterion Revisited

No.	Author(s)	Title
9321	A.L. Bovenberg and S. Smulders	Environmental Quality and Pollution-saving Technological Change in a Two-sector Endogenous Growth Model
9322	K.-E. Wärneryd	The Will to Save Money: an Essay on Economic Psychology
9323	D. Talman, Y. Yamamoto and Z. Yang	The $(2^{n+m+1} - 2)$ -Ray Algorithm: A New Variable Dimension Simplicial Algorithm For Computing Economic Equilibria on $S^n \times R_+^m$
9324	H. Huizinga	The Financing and Taxation of U.S. Direct Investment Abroad
9325	S.C.W. Eijffinger and E. Schaling	Central Bank Independence: Theory and Evidence
9326	T.C. To	Infant Industry Protection with Learning-by-Doing
9327	J.P.J.F. Scheepens	Bankruptcy Litigation and Optimal Debt Contracts
9328	T.C. To	Tariffs, Rent Extraction and Manipulation of Competition
9329	F. de Jong, T. Nijman and A. Röell	A Comparison of the Cost of Trading French Shares on the Paris Bourse and on SEAQ International
9330	H. Huizinga	The Welfare Effects of Individual Retirement Accounts
9331	H. Huizinga	Time Preference and International Tax Competition
9332	V. Feltkamp, A. Koster, A. van den Nouweland, P. Borm and S. Tijs	Linear Production with Transport of Products, Resources and Technology
9333	B. Lauterbach and U. Ben-Zion	Panic Behavior and the Performance of Circuit Breakers: Empirical Evidence
9334	B. Melenberg and A. van Soest	Semi-parametric Estimation of the Sample Selection Model
9335	A.L. Bovenberg and F. van der Ploeg	Green Policies and Public Finance in a Small Open Economy
9336	E. Schaling	On the Economic Independence of the Central Bank and the Persistence of Inflation
9337	G.-J. Otten	Characterizations of a Game Theoretical Cost Allocation Method
9338	M. Gradstein	Provision of Public Goods With Incomplete Information: Decentralization vs. Central Planning
9339	W. Güth and H. Kliemt	Competition or Co-operation
9340	T.C. To	Export Subsidies and Oligopoly with Switching Costs

No.	Author(s)	Title
9341	A. Demirgüç-Kunt and H. Huizinga	Barriers to Portfolio Investments in Emerging Stock Markets
9342	G.J. Almekinders	Theories on the Scope for Foreign Exchange Market Intervention
9343	E.R. van Dam and W.H. Haemers	Eigenvalues and the Diameter of Graphs
9344	H. Carlsson and S. Dasgupta	Noise-Proof Equilibria in Signaling Games
9345	F. van der Ploeg and A.L. Bovenberg	Environmental Policy, Public Goods and the Marginal Cost of Public Funds
9346	J.P.C. Blanc and R.D. van der Mei	The Power-series Algorithm Applied to Polling Systems with a Dormant Server
9347	J.P.C. Blanc	Performance Analysis and Optimization with the Power-series Algorithm
9348	R.M.W.J. Beetsma and F. van der Ploeg	Intramarginal Interventions, Bands and the Pattern of EMS Exchange Rate Distributions
9349	A. Simonovits	Intercohort Heterogeneity and Optimal Social Insurance Systems
9350	R.C. Douven and J.C. Engwerda	Is There Room for Convergence in the E.C.?
9351	F. Vella and M. Verbeek	Estimating and Interpreting Models with Endogenous Treatment Effects: The Relationship Between Competing Estimators of the Union Impact on Wages
9352	C. Meghir and G. Weber	Intertemporal Non-separability or Borrowing Restrictions? A Disaggregate Analysis Using the US CEX Panel
9353	V. Feltkamp	Alternative Axiomatic Characterizations of the Shapley and Banzhaf Values
9354	R.J. de Groof and M.A. van Tuijl	Aspects of Goods Market Integration. A Two-Country-Two-Sector Analysis
9355	Z. Yang	A Simplicial Algorithm for Computing Robust Stationary Points of a Continuous Function on the Unit Simplex
9356	E. van Damme and S. Hurkens	Commitment Robust Equilibria and Endogenous Timing
9357	W. Güth and B. Peleg	On Ring Formation In Auctions
9358	V. Bhaskar	Neutral Stability In Asymmetric Evolutionary Games
9359	F. Vella and M. Verbeek	Estimating and Testing Simultaneous Equation Panel Data Models with Censored Endogenous Variables

No.	Author(s)	Title
9360	W.B. van den Hout and J.P.C. Blanc	The Power-Series Algorithm Extended to the <i>BMAP/PH/1</i> Queue
9361	R. Heuts and J. de Klein	An (s, q) Inventory Model with Stochastic and Interrelated Lead Times
9362	K.-E. Wärmeryd	A Closer Look at Economic Psychology
9363	P.J.-J. Herings	On the Connectedness of the Set of Constrained Equilibria
9364	P.J.-J. Herings	A Note on "Macroeconomic Policy in a Two-Party System as a Repeated Game"
9365	F. van der Ploeg and A. L. Bovenberg	Direct Crowding Out, Optimal Taxation and Pollution Abatement
9366	M. Pradhan	Sector Participation in Labour Supply Models: Preferences or Rationing?
9367	H.G. Bloemen and A. Kapteyn	The Estimation of Utility Consistent Labor Supply Models by Means of Simulated Scores
9368	M.R. Baye, D. Kovenock and C.G. de Vries	The Solution to the Tullock Rent-Seeking Game When $R > 2$: Mixed-Strategy Equilibria and Mean Dissipation Rates
9369	T. van de Klundert and S. Smulders	The Welfare Consequences of Different Regimes of Oligopolistic Competition in a Growing Economy with Firm-Specific Knowledge
9370	G. van der Laan and D. Talman	Intersection Theorems on the Simplotope
9371	S. Muto	Alternating-Move Preplays and $vN - M$ Stable Sets in Two Person Strategic Form Games
9372	S. Muto	Voters' Power in Indirect Voting Systems with Political Parties: the Square Root Effect
9373	S. Smulders and R. Gradus	Pollution Abatement and Long-term Growth
9374	C. Fernandez, J. Osiewalski and M.F.J. Steel	Marginal Equivalence in v -Spherical Models
9375	E. van Damme	Evolutionary Game Theory
9376	P.M. Kort	Pollution Control and the Dynamics of the Firm: the Effects of Market Based Instruments on Optimal Firm Investments
9377	A. L. Bovenberg and F. van der Ploeg	Optimal Taxation, Public Goods and Environmental Policy with Involuntary Unemployment

No.	Author(s)	Title
9378	F. Thuijsman, B. Peleg, M. Amitai & A. Shmida	Automata, Matching and Foraging Behavior of Bees
9379	A. Lejour and H. Verbon	Capital Mobility and Social Insurance in an Integrated Market
9380	C. Fernandez, J. Osiewalski and M. Steel	The Continuous Multivariate Location-Scale Model Revisited: A Tale of Robustness
9381	F. de Jong	Specification, Solution and Estimation of a Discrete Time Target Zone Model of EMS Exchange Rates
9401	J.P.C. Kleijnen and R.Y. Rubinstein	Monte Carlo Sampling and Variance Reduction Techniques
9402	F.C. Drost and B.J.M. Werker	Closing the Garch Gap: Continuous Time Garch Modeling
9403	A. Kapteyn	The Measurement of Household Cost Functions: Revealed Preference Versus Subjective Measures
9404	H.G. Bloemen	Job Search, Search Intensity and Labour Market Transitions: An Empirical Exercise
9405	P.W.J. De Bijl	Moral Hazard and Noisy Information Disclosure
9406	A. De Waegenaere	Redistribution of Risk Through Incomplete Markets with Trading Constraints
9407	A. van den Nouweland, P. Borm, W. van Golstein Brouwers, R. Groot Bruinderink, and S. Tijs	A Game Theoretic Approach to Problems in Telecommunication
9408	A.L. Bovenberg and F. van der Ploeg	Consequences of Environmental Tax Reform for Involuntary Unemployment and Welfare
9409	P. Smit	Arnoldi Type Methods for Eigenvalue Calculation: Theory and Experiments
9410	J. Eichberger and D. Kelsey	Non-additive Beliefs and Game Theory

P.O. BOX 90153, 5000 LE TILBURG, THE NETHERLAND

Bibliotheek K. U. Brabant



17 000 01138432 9